Thermodynamics: Heat Transfer

FIZIKA SPhO Training

September 2025

Contents

1 Notes

Heat transfer occurs mainly via **three methods:** conduction, convection and radiation.

1.1 Conduction

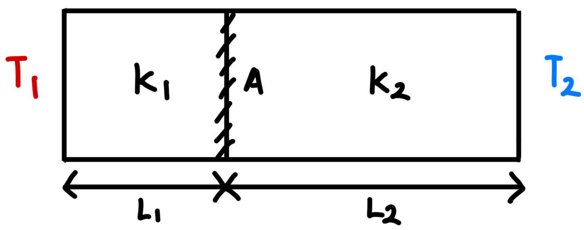
Heat conduction follows **Fourier's Law:**

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -kA\frac{\mathrm{d}T}{\mathrm{d}x} \tag{1}$$

where k is the thermal conductivity (units of $\mathrm{W/m \cdot K}$), and the negative sign indicates that heat flows from hot to cold.

Usually, for conduction problems, we make the assumption that **steady-state** has been reached. This implies that $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is constant throughout the material.

Example 1.1. Two metal slabs of lengths L_1 and L_2 with thermal conductivities k_1 and k_2 respectively are joined together. The area of the face joining them together is A. The two ends of the slabs are placed in contact with two heat reservoirs of temperatures T_1 and $T_2 < T_1$ respectively. Find the net rate of heat flow.



Let the interface temperature be T. Clearly, $T_2 < T < T_1$ at steady state.

Applying the steady-state condition, we have

$$\frac{\mathrm{d}Q}{\mathrm{d}t} \text{ is constant} \quad \Rightarrow \quad k_1A\frac{T_1 - T}{L_1} = k_2A\frac{T - T_2}{L_2} \quad \Rightarrow \quad T = \frac{k_1T_1L_2 + k_2T_2L_1}{k_1L_2 + k_2L_1}$$

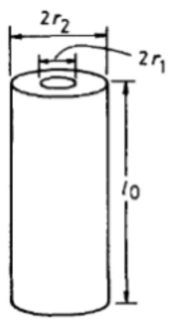
Hence, the net rate of heat flow is

$$P = \left|\frac{\mathrm{d}Q}{\mathrm{d}t}\right| = k_1A\frac{T_1 - T}{L_1} = \frac{k_1k_2A\left(T_1 - T_2\right)}{k_1L_2 + k_2L_1}$$

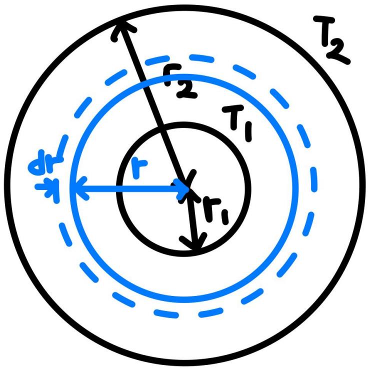
You can see that by setting $k_1 = k_2 = k$, this is equivalent to normal heat conduction through a single metal slab of conductivity k and length $L_1 + L_2$ between temperatures T_1 and T_2, which makes sense.

Some problems may require you to solve Equation (1) as a differential equation, such as the example below.

Example 1.2 (Ricardo)**.** A uniform non-metallic annular cylinder of inner radius r_1, outer radius r_2 and length l_0 is maintained with its inner surface at T_1 and its outer surface at $T_2 < T_1$. Find the radial temperature distribution $T\left(r\right)$ in the cylinder at steady state.



To do this, we split up the annular cylinder into its cylindrical layers:



Then, imposing the steady-state condition:

$$P = \frac{\mathrm{d}Q}{\mathrm{d}t} = \text{constant} \quad \Rightarrow \quad P = kA\frac{\mathrm{d}T}{\mathrm{d}r} = 2\pi k r l_0 \frac{\mathrm{d}T}{\mathrm{d}r} \quad \Rightarrow \quad \frac{\mathrm{d}r}{r} = \frac{2\pi k l_0}{P}\,\mathrm{d}T$$

We can now impose the boundary conditions, which are $T\left(r_1\right) = T_1$ and $T\left(r_2\right) = T_2$:

$$\int_{T_1}^{T_2} \frac{2\pi k l_0}{P}\,\mathrm{d}T = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r} \quad \Rightarrow \quad P = \frac{2\pi k l_0}{\ln\left(\frac{r_2}{r_1}\right)}\left(T_2 - T_1\right)$$

With P, we can now find the general $T\left(r\right)$:

$$\int_{T_1}^{T(r)} \frac{2\pi k l_0}{P}\,\mathrm{d}T = \int_{r_1}^{r} \frac{\mathrm{d}r}{r} \quad \Rightarrow \quad \frac{T\left(r\right) - T_1}{T_2 - T_1} \ln\left(\frac{r_2}{r_1}\right) = \ln\left(\frac{r}{r_1}\right)$$

$$\Rightarrow \quad T\left(r\right) = T_1 + \left(\frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}\right)\left(T_2 - T_1\right)$$

1.2 Convection

Convection follows **Newton's Law of Cooling:**

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = hA\Delta T \tag{2}$$

where h is the heat transfer coefficient (units of $\mathrm{W/m \cdot K}$), and ΔT is the temperature **difference** between the object and its environment.

Example 1.3 (Ricardo)**.** A thin wall of exposed surface area A, heat capacity C and initial temperature T_0 is placed in a room where the temperature of the air is $T_{\text{air}} < T_0$. The heat transfer coefficient between the wall and the air is h. Find the temperature of the wall as a function of time, assuming only convection is significant.

Applying Equation (2) directly, we have

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = hA\left(T - T_{\text{air}}\right) \quad \Rightarrow \quad -C\frac{\mathrm{d}T}{\mathrm{d}t} = hA\left(T - T_{\text{air}}\right)$$

where the negative sign is important, as the wall is *losing* heat to the air.

We can rearrange to get a first-order ODE:

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{hA}{C}T = \frac{hA}{C}T_{\text{air}} \quad \Rightarrow \quad T\left(t\right) = \left(T_0 - T_{\text{air}}\right)\exp\left(-\frac{hA}{C}t\right) + T_{\text{air}}$$

You can check that as $t \to \infty$, $T\left(t\right) \to T_{\text{air}}$, which makes sense as thermal equilibrium will be achieved after a long time.

1.3 Radiation

Radiation follows **Stefan-Boltzmann's Law:**

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = A\varepsilon\sigma T^4 \tag{3}$$

where $0 \le \varepsilon \le 1$ is the emissivity and $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$ is the Stefan-Boltzmann constant. Perfect reflectors have $\varepsilon = 0$ while perfect blackbodies have $\varepsilon = 1$.

Remark. If not stated, in Olympiad problems, it is assumed that $\varepsilon = 1$.

Example 1.4 (IPhO 1992)**.** A satellite is a sphere of diameter D orbiting about the Earth. Suppose the satellite is perfectly black. Ignoring the effect of the Earth, find its temperature T in terms of the temperature of the Sun T_{s}, the radius of the Sun R_{s} and the radius of Earth's orbit around the Sun R.

The power going into the satellite from the Sun is

$$P_{\text{in}} = IA = \left(\frac{P_{\text{s}}}{4\pi R^2}\right)\left(\frac{1}{4}\pi D^2\right) = \left(\frac{4\pi R_{\text{s}}^2\sigma T_{\text{s}}^4}{4\pi R^2}\right)\left(\frac{1}{4}\pi D^2\right) = \frac{\sigma R_{\text{s}}^2 T_{\text{s}}^4 \pi D^2}{4R^2}$$

The power radiated by the satellite is

$$P_{\text{out}} = 4\pi\left(\frac{D}{2}\right)^2\sigma T^4 = \pi D^2\sigma T^4$$

At steady-state, $P_{\text{in}} = P_{\text{out}}$. Hence,

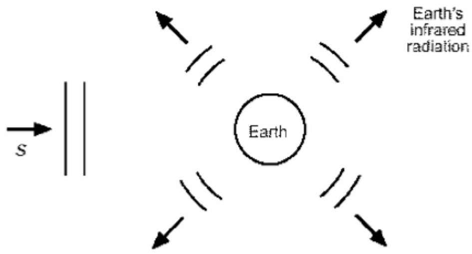
$$\frac{\sigma R_{\text{s}}^2 T_{\text{s}}^4 \pi D^2}{4R^2} = \pi D^2\sigma T^4 \quad \Rightarrow \quad T = T_{\text{s}}\sqrt{\frac{R_{\text{s}}}{2R}}$$

2 Problems

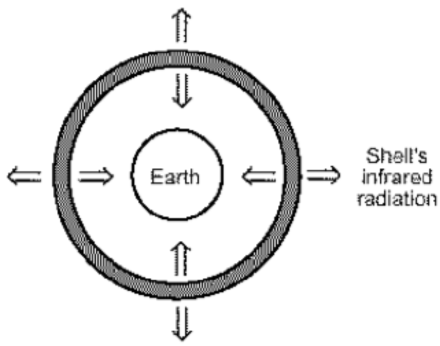
Problems are arranged in roughly increasing difficulty. Take $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\ m^{-2}\,K^{-4}}$ if needed.

Problem 2.1 (SPhO 2002)**.** (i) The interior of the Earth can be roughly modelled as follows: it consists of a molten core 3440 km in radius, a mantle surrounding the core with thickness 2900 km, and a rocky crust with thickness 30 km. Heat is generated from within the core due to the decay of radioactive elements, and conducts upwards to the surface through the mantle and crust. Suppose the core generates 4×10^{13} W of heat and its outer temperature is 3700°C. If the thermal conductivity of the mantle and the crust is $157\,\mathrm{W/m \cdot K}$ and $2.35\,\mathrm{W/m \cdot K}$ respectively, what is the resulting temperature at the surface of the Earth assuming there are no other forms of heat gain or loss?

(ii) You might have found your answer in part (i) unrealistic. Of course, heat also reaches the Earth's surface from the Sun. The Sun delivers radiation to the Earth at a rate of $1400\,\mathrm{W/m^2}$, measured at right angles to the Sun's rays. However, we must account for the fact that 30% of the solar radiation is reflected back out to space without being absorbed. Now, the Earth's surface also radiates infrared radiation back into space. Assuming it radiates like a blackbody, what is the average temperature of the Earth's surface?



(iii) Your answer in part (ii) should still be unrealistic. A more realistic model takes into account the greenhouse warming effect of the atmosphere. This is done by treating the atmosphere as a spherical shell enclosing the Earth. It is transparent to incoming (visible) solar radiation but absorbs some of the outgoing infrared radiation from the Earth's surface. It is thus heated to a certain temperature and radiates its own infrared radiation, **from both the inner and outer surfaces.** Assume that the shell absorbs 75% of the outgoing infrared radiation, and that its (inner or outer) surface area is the same as that of Earth's surface. Calculate the new value for the temperature of the Earth's surface.



Problem 2.2 (Ricardo)**.** Imagine two concentric spheres, the smaller one with radius r_1 and temperature T_1 and the larger one with radius r_2 and temperature T_2. If the space between them is filled with a material of thermal conductivity k and $T_1 > T_2$, find the heat transfer rate.

Problem 2.3 (Kevin Zhou)**.** Consider a sphere of radioactive rock, which constantly produces heat σ per unit volume. The outside of the sphere is held at temperature T_0, the sphere's radius is R, and its thermal conductivity is k. Find the temperature at the centre of the sphere.

Problem 2.4 (IPhO 1996)**.** Two perfectly black surfaces of temperatures T_h and $T_\mathrm{l} < T_\mathrm{h}$ are placed parallel to each other in a vacuum, and the net heat flux from the hotter surface to the colder one is P. Now, suppose that 2 parallel, thermally insulating, perfectly black plates are placed in between them. This shielding reduces the heat flux to P'. Find P' in terms of P. Can you generalise your answer to N such plates between them?

Problem 2.5 (USAPhO 2016, Problem A4)**.** An excellent heat conduction problem that involves some casework.

Problem 2.6. Consider two walls of identical size with different emissivities ε_1 and ε_2 facing each other. If both walls have the same temperature T, verify quantitatively that there is no net heat flow from one to another.

3 Advanced Problems

These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

Problem 3.1 (EuPhO 2019, T1)**.** An excellent problem on an interesting atmospheric phenomenon.

Problem 3.2 (Physics Cup 2022, Problem 5)**.** An excellent problem linking radiation and heat engines. A simpler variation of this problem is USAPhO 2019, Problem A2.